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On the Sum of Proximinal Subspaces

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1. INTRODUCTION

A subset M of a Banach space X is said to be proximal if, for every x in X, $\inf\{\|x-m\|: m \in M\}$ is attained. The following was raised by E. W. Cheney [1].

PROBLEM. If U and V are proximinal subspaces in a Banach space X and if $U+V$ is closed, does it follow that $U+V$ is proximinal?

We give a negative answer to this problem. However, we show that if V is reflexive and $U \cap V$ is finite dimensional (in particular, if V is finite dimensional), then the answer is positive.

2. EXAMPLE

If x is an element of c_0, we denote its nth coordinate by x_n. For any positive integer n let $e^{(n)} = (0, ..., 0, 1, 0, ...) \in c_0$ (the 1 in the nth place).

Let

$$U = \left\{ x \in c_0 \text{: for all } n,\ x_{2n} = 0; \sum 2^{-n} x_{2n-1} = 0 \right\}$$

$$V = \left\{ x \in c_0 \text{: for all } n,\ x_{2n-1} = 0; \sum 2^{-n} x_{2n} = 0 \right\}$$

$$M = U + V = \left\{ x \in c_0 \text{: } \sum 2^{-n} x_{2n-1} = 0 = \sum 2^{-n} x_{2n} = 0 \right\}$$

and

$$X = M + \{\lambda f \text{: } \lambda \text{ scalar}\} \subset c_0$$

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where

$$f = e^{(1)} + e^{(2)} = (1, 1, 0, 0, 0, \ldots).$$

Then U and V are proximinal in X, while $M = U + V$ is not.

Proof. (a) We show first that U is proximinal in X. Given $y \in U$, $z \in V$ and λ a scalar, we must prove that there is $u \in U$ such that

$$d(y + z + \lambda f, U) = \|y + z + \lambda f - u\|.$$

We may clearly assume that $y = 0$ since $y \in U$. Further, it is enough to prove such u exists when $\lambda = 1$ since when $\lambda = 0$ choosing $u = 0$ is possible since $d(z, U) = \|z\|$ (for z is supported on the even integers and all elements of U are supported on the odd ones), and when $\lambda \neq 0$ we use homogeneity.

Thus, we need to show only that if $z \in V$ there is $u \in U$ such that

$$d(z + f, U) = \|z + f - u\|.$$

Now,

$$d(z + f, U) = \max\{\|z + e^{(2)}\|, d(e^{(1)}, U)\}. \tag{1}$$

For every n the sequence

$$g^{(n)} = \tfrac{1}{2}(1 - 2^{-n}, 0, -1, 0, -1, 0, -1, \ldots, 0, -1, 0, 0, \ldots)$$

with -1 appearing n times is an element of U and

$$\|e^{(1)} - g^{(n)}\| \to \tfrac{1}{2}.$$

Thus

$$d(e^{(1)}, U) \leqslant \tfrac{1}{2}. \tag{2}$$

We now claim that

$$\|z + e^{(2)}\| > \tfrac{1}{2}. \tag{3}$$

Assume to the contrary that $\|z + e^{(2)}\| \leqslant \frac{1}{2}$; then $|z_2| \geqslant \frac{1}{2}$ and $|z_k| \leqslant \frac{1}{2}$ for all $k > 2$.

Since $\sum 2^{-n} z_{2n} = 0$, we get

$$\tfrac{1}{4} \leqslant \tfrac{1}{2}|z_2| = \left|\sum_{n \geqslant 2} 2^{-n} z_{2n}\right| \leqslant \sum_{n \geqslant 2} 2^{-n} |z_{2n}| \leqslant \tfrac{1}{2} \sum_{n \geqslant 2} 2^{-n} = \tfrac{1}{4}.$$

Hence we must have equalities throughout the last string of inequalities, which can happen only if $|z_{2n}| = \frac{1}{2}$ for all n. But this contradicts z being in c_0.

Thus we have

$$\|z+e^{(2)}\| > \tfrac{1}{2} \geqslant d(e^{(1)}, U). \tag{4}$$

This together with (1) give

$$d(z+f, U) = \|z+e^{(2)}\|.$$

By (4) we have some $u \in U$ such that

$$\|e^{(1)} - u\| = \|z+e^{(2)}\|.$$

This yields

$$\|z+f-u\| = \|z+e^{(2)}\| = d(z+f, U).$$

(b) The proof that V is proximinal in X is similar (also, there is an isometry on X that interchanges U and V).

(c) Finally, to prove that M is not proximinal note that we have shown already that $\|z+e^{(2)}\| > \frac{1}{2}$ for every $z \in V$. If x is any element in M then $x = y+z$ for suitable $y \in U$, $z \in V$ and

$$\|x+f\| \geqslant \|z+e^{(2)}\| > \tfrac{1}{2}.$$

However,

$$d(f, M) = \tfrac{1}{2}$$

since

$$\|f - h^{(n)}\| \to \tfrac{1}{2}$$

when

$$h^{(n)} = \tfrac{1}{2}(1-2^{-n}, 1-2^{-n}, -1, -1, -1, ..., -1, 0, 0, ...)$$

while $h^{(n)} \in X$ when -1 appears the right number of times $(2n)$.

Therefore X is not proximinal.

Remark. In fact $X = \{x \in c_0 : \sum 2^{-n}(x_{2n} - x_{2n-1}) = 0\}$, a hyperplane in c_0.

3. Some Positive Results

THEOREM. *Let F and G be subspaces of a Banach space X. Assume that F is proximinal, G is reflexive, $F \cap G$ is finite dimensional and $F+G$ is closed. Then $F+G$ is proximinal.*

Proof. Assume first that $F \cap G$ is $\{0\}$. Let $x_0 \in X$. There are sequences $\{f_n\}$ in F and $\{g_n\}$ in G such that

$$\|x_0 - (f_n + g_n)\| \to d(x_0, F+G).$$

Clearly $\{f_n + g_n\}$ is bounded. The linear projection P from $F+G$ onto G vanishing on F (i.e., $P(f+g) = g$ whenever f is in F and g is in G) is bounded (by the Closed Graph Theorem). Thus $\{g_n\}$ is also bounded and so is $\{f_n\}$.

Since G is reflexive $\{g_n\}$ has a weakly convergent subsequence. By passing to a subsequence we may assume that $g_n \to^{\omega} g_0$ for some $g_0 \in G$. Thus, there is a sequence of convex combinations

$$\tilde{g}_n = \sum_{i \in I_n} \lambda_i g_i$$

(where $I_n = \{i\colon p_n < i \leqslant p_{n+1}\}$, p_n an increasing sequence of integers, $\lambda_i \geqslant 0$ and $\sum_{i \in I_n} \lambda_i = 1$) such that $\|\tilde{g}_n - g_0\| \to 0$.

Denote

$$\tilde{f}_n = \sum_{i \in I_n} \lambda_i f_i$$

then

$$\|x_0 - \tilde{f}_n - \tilde{g}_n\| = \left\| \sum_{i \in I_n} \lambda_i (x_0 - f_i - g_i) \right\| \leqslant \sum_{i \in I_n} \lambda_i \, \|x_0 - f_i - g_i\|.$$

Hence

$$\|x_0 - \tilde{f}_n - \tilde{g}_n\| \to d(x_0, F+G).$$

Also

$$\|x_0 - (\tilde{f}_n + g_0)\| \leqslant \|x_0 - (\tilde{f}_n + \tilde{g}_n)\| + \|\tilde{g}_n - g_0\|$$

which gives

$$\|x_0 - (\tilde{f}_n + g_0)\| \to d(x_0, F+G).$$

Let f_0 be the nearest element to $x_0 - g_0$ in F, then

$$\|x_0 - (f_0 + g_0)\| \leqslant \|x_0 - \tilde{f}_n - g_0\| \to d(x_0, F+G).$$

This means that

$$\|x_0 - (f_0 + g_0)\| = d(x_0, F + G)$$

and thus $F + G$ is proximinal.

Finally, if the intersection of F and G is not $\{0\}$ but a finite dimensional subspace, we can find a closed subspace G_1 of G (hence reflexive) such that $F + G = F + G_1$ and with a trivial intersection of F and G_1.

COROLLARY. *Let F and G be subspaces of a Banach space X, with F proximinal and G finite dimensional. Then $F + G$ is proximinal.*

Proof. Since G is finite dimensional and F must be closed, $F + G$ is closed.

REFERENCE

1. "Canad. Math. Soc. Conf. Proc. 3," 1983, p. 391.